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Graphs In Which Upper Strong Efficient Domination Number Equals The Independence Number

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ABSTRACT: *Let G = (V, E) be a simple graph. A subset S of V(G) is called a strong (weak) efficient dominating set of G if for every $v \in V(G)$, $|N_s[v] \cap S| = 1$ ($|N_w[v] \cap S| = 1$), where $N_s(v) = \{ u \in V(G): uv \in E(G), \deg u \geq \deg v \}$ and $N_w(v) = \{ u \in V(G) : uv \in E(G), \deg v \geq \deg u\}$, $N_s[v] = N_s(v) \cup \{v\}$, ($N_w[v] = N_w(v) \cup \{v\}$). The minimum cardinality of a strong (weak) efficient dominating set is called the strong (weak) efficient domination number of G and is denoted by $\gamma_{se}(G)$ ($\gamma_{we}(G)$). A graph is strong efficient if there exists a strong efficient dominating set of G. The graphs which have full degree vertex admit strong efficient dominating set. Not all graphs admit strong efficient dominating sets. The complete bipartite graph $K_{m,n}$, $m,n \geq 2$, doesn't admit strong efficient dominating set. A subset S of V(G) is called an efficient dominating set of G if $/N[v] \cap S/ = 1$ for all vertices $v \in V(G)$. If G has an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number $\gamma(G)$ [1]. In particular all efficient dominating sets of G have the same cardinality. But this is not true in the case of strong efficient domination. The maximum cardinality of any strong efficient dominating set of G is called the upper strong efficient domination number of G and is denoted by $\Gamma_{se}(G)$. A set of vertices of G is said to be independent if no two of them are adjacent. The maximum number of vertices in any independent set of G is called the independence number of G and is denoted by $\beta_0(G)$. Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. The Cartesian product $G_1 \square G_2$ is defined to be the graph whose vertex set is $V_1 \times V_2$ and edge set is $\{(u_1,v_1),(u_2,v_2) / \text{ either } u_1 = u_2 \text{ and } v_1v_2 \in E_2 \text{ or } v_1 = v_2 \text{ and } u_1u_2 \in E_1\}$. In this paper, we introduce the concept of perfect graphs of upper strong efficient domination and maximum independent sets. That is, graphs G in which $\Gamma_{se}(G) = \beta_0(G)$. A study of such graphs is made. Also strong efficient domination in product graphs is discussed.*

KEY WORDS: *Strong efficient domination number, strong dominating set, independent strong dominating set.*
AMS Subject Classification (2010): *05C69.*

I. INTRODUCTION

Throughout this paper, we consider finite, undirected, simple graphs. Let G = (V,E) be a simple graph. The degree of any vertex u in G is the number of edges incident with u and is denoted by deg u. The minimum and maximum degree of a vertex is denoted by $\delta(G)$ and $\Delta(G)$ respectively. A vertex of degree 0 in G is called an isolated vertex and a vertex of degree 1 is called a pendant vertex. A subset S of V(G) of a graph G is called a dominating set if every vertex in V(G) \ S is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. E. Sampathkumar and L.Pushpalatha introduced the concepts of strong and weak domination in graphs [8]. A subset S of V(G) is called a strong dominating set of G if for every $v \in V - S$ there exists $u \in S$ such that u and v are adjacent and $\deg u \geq \deg v$. The strong domination number $\gamma_s(G)$ is the minimum cardinality of a strong dominating set of G and the strong domination number $\Gamma_s(G)$ is the maximum cardinality of a strong dominating set of G. The independent strong domination number $i_s(G)$ is the minimum cardinality of an independent strong dominating set of G. The strong domination and independent numbers were studied in [3, 4, 6, 7, 8]. The upper independent strong domination number $\beta_s(G)$ is the maximum cardinality of an independent strong dominating set of G. The maximum number of vertices in any independent set of G is called the independence number of G and it is denoted by $\beta_0(G)$. The inequality chain $\gamma_s(G) \leq i_s(G) \leq \gamma_{se}(G) \leq \beta_s(G) \leq \Gamma_s(G) \leq \beta_0(G)$.holds in any graph admitting strong efficient dominating set. It has been proved in [5], that there are graphs in which strict inequality holds in the chain. In this paper, we introduce the concept of perfect graphs of upper strong efficient domination and maximum independent sets. That is, graphs G in which $\Gamma_{se}(G) = \beta_0(G)$. A study of such graphs is made. Strong efficient domination in products of graphs is also discussed. For all graph theoretic terminologies and notations, we follow Harary [2].

II. MAIN RESULTS

Definition 2.1: Let G = (V, E) be a simple graph. A subset S of V(G) is called a strong (weak) efficient dominating set of G if for every $v \in V(G)$, $|N_s[v] \cap S| = 1$ ($|N_w[v] \cap S| = 1$), where $N_s(v) = \{ u \in V(G) : uv \in E(G), \deg u \geq \deg v \}$ and $N_w(v) = \{ u \in V(G) : uv \in E(G), \deg v \geq \deg u\}$, $N_s[v] = N_s(v) \cup \{v\}$ ($N_w[v] = N_w(v) \cup \{v\}$). The minimum cardinality of a strong (weak) efficient dominating set of G is called the strong (weak) efficient domination number of G and is denoted by $\gamma_{se}(G)$ ($\gamma_{we}(G)$). A graph G is strong efficient if there exists a strong efficient dominating set of G.

Definition 2.2: A subset S of V(G) which is strong efficient and whose cardinality is $\beta_0(G)$ is called independence number preserving strong efficient dominating set of G.

Remark 2.3: Since any strong efficient dominating set is independent, $\Gamma_{se}(G) \leq \beta_0(G)$. Therefore if there exists an independence number preserving strong efficient dominating set of G, then such a set has cardinality $\Gamma_{se}(G)$ and $\Gamma_{se}(G) = \beta_0(G)$.

Remark 2.4: Suppose there exists a graph G such that $\gamma_{se}(G) = \beta_0(G)$. Then $\gamma_{se}(G) = \Gamma_{se}(G) = \beta_0(G)$.

Example 2.5: Let $G = P_5$. $\{v_1, v_3, v_5\}$ is the unique strong efficient dominating set which is an independent set of maximum cardinality also. Therefore $\gamma_{se}(G) = \beta_0(G)$.

Theorem 2.6: Let G_n ($n \geq 3$) be a graph obtained by attaching a vertex of K_n with the central vertex of $K_{1, n-1}$ by an edge. Then $\Gamma_{se}(G_n) = \beta_0(G_n)$.

Proof: Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and $V(K_{1, n-1}) = \{u, u_1, u_2, \dots, u_{n-1}\}$. In G_n, u is attached with any vertex, say v_i, of K_n. Therefore $\deg u = \deg v_i = n = \Delta(G_n)$. For j = 1 to n consider the sets $S_j = \{u, v_j\}$, $j \neq i$. Then u strongly dominates $u_1, u_2, \dots, u_{n-1}$ and v_i. The vertex v_j strongly dominates all the vertices of K_n other than v_i, since $\deg v_j < \deg v_i$, $j \neq i$. Hence S_j, j = 1 to n are strong efficient dominating sets of G_n and $|S_j| = 2$. Consider the set $S = \{v_i, u_1, u_2, \dots, u_{n-1}\}$. v_1 strongly dominates the vertices of K_n and u. The vertices $u_1, u_2, \dots, u_{n-1}$ dominate themselves. Clearly S is also a strong efficient dominating set of G_n and $|S| = n$. The sets S_j s and S are the independent and S is of maximum cardinality. Since any strong efficient dominating set of G_n must contain either u or v_1, no other strong efficient dominating sets exist. Therefore $\gamma_{se}(G_n) = 2$ and $\Gamma_{se}(G_n) = \beta_0(G_n) = n$ where ($n \geq 3$). That is S is the independence number preserving strong efficient dominating set of G_n. ■

Remark 2.7: In the above theorem, when n = 1, G_n is P_2, $\gamma_{se}(G_n) = \beta_0(G_n) = 1$.
When n = 2, G_n is P_4, $\gamma_{se}(G_n) = \beta_0(G_n) = 2$.

Theorem 2.8: A graph G doesn′t admit a strong efficient dominating set if G has exactly two vertices of degree $\Delta(G)$ such that d (u,v) = 2.

Proof: Let G = (V, E) be a simple graph. Suppose G admits a strong efficient dominating set S. Let u, v be the vertices of degree $\Delta(G)$ such that d (u, v) = 2. Therefore there exists a vertex $w \in V(G)$ adjacent with u and v. Since u and v are not adjacent, $u, v \in S$. Therefore $|N_s[w] \cap S| \geq 2$, a contradiction. Therefore G doesn't admit a strong efficient dominating set if G has exactly two vertices of degree $\Delta(G)$ such that d (u, v) = 2. ■

Theorem 2.9: Let $G = P_k$ where k = 2n + 1, n > 2. Attach a vertex u with G and make u adjacent with every vertex of an independent set $\{v_2, v_4, \dots v_{2n}\}$ of G. The resulting graph H is strong efficient and $\gamma_{se}(H) = \beta_0(H)$.

Proof: Let $V(G) = \{v_1, v_2, v_3, \dots, v_{2n+1}\}$. In H, u is made adjacent with $v_2, v_4, \dots, v_{2n}$. Since $\deg u = n = \Delta(G)$, u is the only maximum degree vertex of H. Since $\deg v_i > \deg v_j$ for all $i \in (2, 4, \dots, 2n)$ and $j \in (1, 3, 5, \dots, 2n+1)$. v_j cannot strongly dominate v_i. Therefore $T = \{u, v_1, v_3, v_5, \dots v_{2n-1}, v_{2n+1}\}$ is the unique strong efficient dominating set of H and T is also the maximum independent set of H. Therefore H is strong efficient and $\gamma_{se}(H) = \beta_0(H)$. ■

Remark 2.10: **(i)** When n = 2, G is P_5. If u is made adjacent with an independent set $\{v_2, v_4\}$ of G, then v_2 and v_4 are the only maximum degree vertices at a distance 2 in the resulting graph is H. By theorem 2.8, H is non strong efficient.
(ii) When n = 1, G is P_3. u is made adjacent with an independent set $\{v_2\}$. In the resulting graph H, v_2 is the full degree vertex of H and hence $\gamma_{se}(H) = 1$. But $\beta_0(H) = 3$. Therefore $\gamma_{se}(H) \neq \beta_0(H)$.

Theorem 2.11: Let $G = C_{2n}$, $n \geq 3$. Attach a vertex u with G and make u adjacent with every vertex of a maximum independent set of G. Then the resulting graph H is strong efficient and $\gamma_{se}(H) = \beta_0(H)$.

Proof: Let $V(G) = \{v_1, v_2, \dots, v_{2n}\}$. Let the new vertex u be adjacent with $v_2, v_4, .. v_{2n}$. Therefore u is the only maximum degree vertex of the resulting graph H. The set $S = \{v_1, v_3, v_5, ... v_{2n-1}\}$ of non neighbours of u is independent and for any v_i belongs to S, deg v_i is less than that of any vertex of N[u]. Therefore $T = \{u, v_1, v_3, \dots, v_{2n-1}\}$ is the unique strong efficient dominating set of H and T is an independent set of maximum cardinality. Hence $\gamma_{se}(H) = \beta_0(H) = n + 1$. ∎

Remark 2. 12: When n = 2, $G = C_4$. If u is made adjacent either with v_2, v_4 or with v_1, v_3, then the resulting graph H is non strong efficient, since distance between the maximum degree vertices is 2, by theorem 2.8.

Definition 2.13 [2]: A rooted tree has one point, its root v distinguished from the others. A rooted tree with root of degree n can be regarded as a configuration whose figures are the n rooted trees obtained on removing the root. G – v are the constituent rooted trees.

Theorem 2.14: Let u be the root of a rooted tree G of degree n, $n \geq 3$. Let one of the constituent rooted tree be $K_{1, n-1}$ and remaining be $K_{1,r}$ where $r < n - 1$. Then G is strong efficient and $\Gamma_{se}(G) = \beta_0(G)$.

Proof: Let $v_1, v_2, \dots, v_n$ be the vertices adjacent with u. Without loss of generality, let deg $v_1 = n - 1$ and deg $v_i = n - 2$, $2 \leq i \leq n$. Therefore u and v_1 are the only maximum degree vertices. Let $v_{11}, v_{12}, \dots, v_{1n-1}$ be the vertices adjacent with v_1, and $v_{i1}, v_{i2}, \dots, v_{in-2}$ be the vertices adjacent with v_i, $2 \leq i \leq n$. u strongly dominates $v_1, v_2, \dots, v_n$, and $v_{11}, v_{12}, \dots, v_{1n-1}$, $v_{i1}, v_{i2}, \dots, v_{in-2}$, $2 \leq i \leq n$ are independent of u. Let $S_1 = \{v_1, v_2, \dots, v_n\}$ and $S_2 = \{u, v_{11}, v_{12}, \dots, v_{1n-1}, v_{21}, v_{22}, \dots, v_{2(n-2)}, \dots, v_{n1}, v_{n2}, \dots, v_{n(n-2)}\}$. Clearly S_1 and S_2 are strong efficient dominating sets. No other strong efficient dominating set without u or v_1 exist. $|S_1| = n$ and $|S_2| = 1 + (n - 1) + (n - 1)(n - 2) = n^2 - 2n + 2$, where $n \geq 3$. $\gamma_{se}(G) = n$ and $\Gamma_{se}(G) = n^2 - 2n + 2$, and S_2 is the maximum independent set of G. Thus $\Gamma_{se}(G) = \beta_0(G)$. Therefore S_2 is the independence number preserving strong efficient dominating set of G. ∎

III. STRONG EFFICIENT DOMINATION IN PRODUCT GRAPHS.

Definition 3.1: Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. The Cartesian product $G_1 \square G_2$ is defined to be the graph whose vertex set is $V_1 \times V_2$ and edge set is $\{(u_1, v_1), (u_2, v_2)$ / either $u_1 = u_2$ and $v_1v_2 \in E_2$ or $v_1 = v_2$ and $u_1u_2 \in E_1\}$.

Theorem 3.2: $P_3 \square P_3$ is strong efficient.

Proof: Let $V(P_3) = \{u_1, u_2, u_3\}$. $V(P_3) = \{v_1, v_2, v_3\}$. (u_2, v_2) is the only maximum degree vertex of $P_3 \square P_3$. (u_2, v_2) strongly uniquely dominates (u_1, v_2), (u_2, v_1), (u_2, v_3), (u_3, v_2). The vertices (u_1, v_1), (u_1, v_3), (u_3, v_1), (u_3, v_3) are independent of (u_2, v_2). Hence $\{(u_2, v_2), (u_1, v_1), (u_1, v_3), (u_3, v_1), (u_3, v_3)\}$ is the unique γ_{se}- set of G. Therefore $\gamma_{se}(P_3 \square P_3) = 5$. ∎

Theorem 3.3: $P_2 \square P_3$ is not strong efficient.

Proof: Let $V(P_2) = \{u_1, u_2\}$. $V(P_3) = \{v_1, v_2, v_3\}$. Suppose S is a strong efficient dominating set of G. Either (u_1, v_2) belongs to S or (u_2, v_2) belongs to S, since they are the only maximum degree vertices. Suppose (u_1, v_2) belongs to S. (Proof is similar if (u_2, v_2) belongs to S). In this case (u_1, v_1), (u_1, v_3), (u_2, v_2) are strongly dominated by (u_1, v_2) in S. Therefore (u_2, v_1) belongs to S. Then (u_1, v_1) is strongly dominated by (u_1, v_2) and (u_2, v_1) in S, a contradiction. Therefore $P_2 \square P_3$ is not strong efficient. ∎

Theorem 3.4: $P_2 \square P_n$ is not strong efficient for any $n \geq 2$.

Proof: For n = 3, the proof is given in the previous theorem. Suppose $n \geq 4$. Let $V(P_2) = \{u_1, u_2\}$. $V(P_n) = \{v_1, v_2, ..., v_n\}$. (u_1, v_1), (u_2, v_1), (u_1, v_n) and (u_2, v_n) are of degree two and all the other vertices of degree three. Suppose S is a strong efficient dominating set of G. Then (u_1, v_2) belongs to S or (u_1, v_3) belongs to S or (u_2, v_2) belongs to S. If (u_1, v_2) belongs to S, then (u_2, v_1) belongs to S leading to a contradiction. If (u_1, v_3) belongs to S, then (u_2, v_2) belongs to S leading to a contradiction. If (u_2, v_2) belongs to S, then (u_1, v_1) belongs to S leading to a contradiction. For n = 2, $P_2 \square P_2 = C_4$, which is not strong efficient. Therefore $P_2 \square P_n$ is not strong efficient for any $n \geq 2$. ∎

Theorem 3.5: $K_{1,n} \square P_3$ is strong efficient and $\gamma_{se}(K_{1,n} \square P_3) = 2n + 1$, $n \geq 1$.

Proof: Let $V(K_{1,n}) = \{u_1, u_2, ..., u_{n+1}\}$. $V(P_3) = \{v_1, v_2, v_3\}$. $V(K_{1,n} \square P_3) = \{(u_i, v_j)/1 \leq i \leq n+1, 1 \leq j \leq 3\}$. $\{(u_1, v_2), (u_2, v_1), (u_3, v_1), \ldots, (u_{n+1}, v_1), (u_2, v_3), (u_3, v_3), \ldots, (u_{n+1}, v_3)\}$ is the unique γ_{se} – set of $K_{1,n} \square P_3$. Therefore $\gamma_{se}(K_{1,n} \square P_3) = 2n + 1$. (Any γ_{se} – set of a graph G must contain the vertex with maximum degree $\Delta(G)$ if it is unique). If S is a γ_{se} – set of $K_{1,n} \square P_3$, then (u_1, v_2) which is of degree n+2 belongs to S. No neighbour of (u_1, v_2) belongs to S, since S is independent. Hence (u_1, v_1), (u_1, v_3), (u_i, v_2), $2 \leq i \leq n+1$, do not belong to S and they are strongly dominated by (u_1, v_2). The remaining 2n vertices in $K_{1,n} \square P_3$ are independent and hence any γ_{se} – set must contain all these 2n vertices. Therefore γ_{se} – set of $K_{1,n} \square P_3$ is unique. ∎

Theorem 3.6: $K_{1,n} \square P_m$ is not strong efficient if $m \neq 3$.

Proof Case (i): Let $m \geq 4$. Let $V(K_{1,n}) = \{u_1, u_2, \ldots, u_{n+1}\}$ and $V(P_m) = \{v_1, v_2, \ldots, v_m\}$. (u_1, v_1) and (u_1, v_m) are of degree n+1. $(u_1, v_2), \ldots, (u_1, v_{m-1})$ are of degree n+2. Let S be a strong efficient dominating set of $G = K_{1,n} \square P_m$. If (u_1, v_2) belongs to S then (u_1, v_4) does not belong to S. (Since otherwise (u_1, v_3) will be strongly dominated by 2 vertices of S) (u_1, v_3) also does not belong to S, since it is adjacent with (u_1, v_2). Therefore (u_2, v_3) belongs to S. In this case, (u_2, v_2) is strongly dominated by (u_1, v_2) and (u_2, v_3), a contradiction. Therefore $(u_1, v_2) \notin S$. Therefore $(u_1, v_3) \in S$. (Since (u_1, v_2) is strongly dominated only by (u_1, v_3)). Therefore $(u_i, v_3) \notin S$, $2 \leq i \leq n+1$. (u_2, v_2) belongs to S, since (u_2,v_2) is strongly dominated by (u_1, v_2) and (u_2, v_3) which do not belong to S. In this case (u_2, v_3) is strongly dominated by two vertices of S namely (u_2, v_2) and (u_1, v_3), a contradiction. Therefore $K_{1,n} \square P_m$ is not strong efficient.

Case (ii): Let m = 2. $V(P_2) = \{v_1, v_2\}$. (u_1, v_1) and (u_1, v_2) are of degree n+1 and the remaining vertices are of degree two. Suppose $K_{1,n} \square P_2$ is strong efficient. Let S be a strong efficient dominating set of $K_{1,n} \square P_2$. Then S must contain either (u_1, v_1) or (u_1, v_2). If $(u_1, v_1) \in S$ then (u_1, v_1) strongly dominates (u_i, v_1), $i = 2, 3, \ldots, n+1$ and (u_1, v_2). Then (u_i, v_2), $i = 2, 3, \ldots, n+1$ belong to S, since they are independent. Hence $|N_s[(u_i, v_1)] \cap S| = |\{(u_1, v_1), (u_i, v_2)\}| = 2 > 1$ for all $i = 2, 3, \ldots, n+1$, a contradiction. Proof is similar if $(u_1, v_2) \in S$. Therefore $K_{1,n} \square P_2$ is not strong efficient. ∎

Theorem 3.7: $C_{4n} \square P_2$, $n \in N$ is strong efficient.

Proof: Let $V(C_{4n}) = \{u_1, u_2, \ldots, u_n\}$ and $V(P_2) = \{v_1, v_2\}$.
$\deg(u_i, v_1) = \deg(u_i, v_2) = 3$ for all $i = 1, 2, \ldots, 4n$.
$S_1 = \{(u_1, v_1), (u_3, v_2), (u_5, v_1), (u_7, v_2), (u_9, v_1), (u_{11}, v_2), \ldots, (u_{4n-3}, v_1), (u_{4n-1}, v_2)\}$
$S_2 = \{(u_2, v_1), (u_4, v_2), (u_6, v_1), (u_8, v_2), (u_{10}, v_1), (u_{12}, v_2), \ldots, (u_{4n-2}, v_1), (u_{4n}, v_2)\}$
$S_3 = \{(u_3, v_1), (u_5, v_2), (u_7, v_1), (u_9, v_2), (u_{11}, v_1), (u_{13}, v_2), \ldots, (u_{4n-1}, v_1), (u_{4n-3}, v_2), (u_1, v_2)\}$ and $S_4 = \{(u_4, v_1), (u_6, v_2), (u_8, v_1), (u_{10}, v_2), (u_{12}, v_1), \ldots, (u_{4n}, v_1), (u_{4n-2}, v_2), (u_2, v_2)\}$. S_1, S_2, S_3 and S_4 are the γ_{se} – sets of $C_{4n} \square P_2$, $n \in N$. Therefore $C_{4n} \square P_2$, $n \in N$ is strong efficient. ∎

Theorem 3.8: $C_m \square P_2$, $m \neq 4n$, $n \in N$ is not strong efficient.

Proof: Case (i): $m = 4n+1$, $n \in N$. $V(C_m) = \{u_1, u_2, \ldots, u_{4n+1}\}$ and $V(P_2) = \{v_1, v_2\}$.
Suppose S is a strong efficient dominating set of $C_{4n+1} \square P_2$. $C_{4n+1} \square P_2$ is a 3 – regular graph. If (u_1, v_1) belongs to S then (u_1, v_2), (u_2, v_1) and (u_{4n+1}, v_1) do not belong to S. (u_2, v_2) does not belong to S, since otherwise (u_1, v_2) is strongly dominated by two elements (u_1, v_1) and (u_2, v_2) of S. Similarly (u_3, v_1) does not belong to S. Therefore (u_3, v_2) belongs to S. Proceeding like this we get (u_5, v_1), (u_7, v_2), …, (u_{4n-1}, v_2) belong to S. (u_{4n-1}, v_2) strongly dominates (u_{4n-1}, v_1), (u_{4n-2}, v_2) and (u_{4n}, v_2). The vertices (u_{4n}, v_1), (u_{4n+1}, v_2) are not strongly dominated by any vertex of S and also they are non adjacent vertices. Therefore (u_{4n}, v_1) and (u_{4n+1}, v_2) belong to S. But (u_{4n}, v_2) is strongly dominated by three elements of S namely (u_{4n}, v_1), (u_{4n+1}, v_2) and (u_{4n-1}, v_2), a contradiction. The case is similar if any other (u_i, v_1) or (u_i, v_2), $1 \leq i \leq 4n+1$, belongs to S. Hence $C_{4n+1} \square P_2$ is not strong efficient.

Case (ii): $m = 4n+2$, $n \in N$. $V(C_m) = \{u_1, u_2, \ldots, u_{4n+1}, u_{4n+2}\}$.
Suppose S is a strong efficient dominating set of $C_{4n+2} \square P_2$. If (u_1, v_1) belongs to S, then as in case (i), (u_3, v_2), (u_5, v_1), (u_7, v_2), …, (u_{4n-1}, v_2), (u_{4n+1}, v_1) belong to S. But (u_{4n+2}, v_1) is strongly dominated by two elements (u_{4n+1}, v_1) and (u_1, v_1) of S, a contradiction and (u_{4n+2}, v_2) is not dominated by any element of S. Also (u_{4n+2}, v_2) does not belong to S, since otherwise (u_{4n+2}, v_1) is strongly dominated by three elements namely (u_{4n+1}, v_1),

(u_{4n+2}, v_2) and (u_1, v_1) of S, a contradiction. The case is similar if any other (u_i, v_1) or (u_i, v_2), $1 \le i \le 4n+2$, belongs to S. Therefore $C_{4n+2} \square P_2$ is not strong efficient.

Case (iii): $m = 4n+3$, $n \in N$. $V(C_m) = \{u_1, u_2, \ldots, u_{4n+1}, u_{4n+2}, u_{4n+3}\}$.
Suppose S is a strong efficient dominating set of $C_{4n+3} \square P_2$. If (u_1, v_1) belongs to S, then as in case (i), (u_3, v_2), (u_5, v_1), (u_7, v_2), …., (u_{4n+1}, v_1), (u_{4n+3}, v_2) belong to S. (u_{4n+3}, v_1) is strongly dominated by two elements (u_1, v_1) and (u_{4n+3}, v_2) of S, a contradiction. The case is similar if any other (u_i, v_1) or (u_i, v_2), $1 \le i \le 4n+3$, belong to S. Hence $C_{4n+3} \square P_2$ is not strong efficient. ■

Remark 3.9: $C_3 \square P_2$ is not strong efficient.
Proof: Let $V(C_3) = \{u_1, u_2, u_3\}$. $V(P_2) = \{v_1, v_2\}$. Suppose S is a strong efficient dominating set of $C_3 \square P_2$. deg $(u_i, v_j) = 3$ for all i, j. Suppose (u_1, v_1) belongs to S. It strongly dominates (u_2, v_1), (u_3, v_1) and (u_1, v_2). If either (u_2, v_2) or (u_3, v_2) belongs to S, leading to contradiction. Therefore they do not belong to S, a contradiction. Similarly we get a contradiction if any (u_i, v_j) belongs to S. Therefore $C_3 \square P_2$ is not strong efficient. ■

Theorem 3.10: $K_m \square P_n$, $n, m \ge 1$ is not strong efficient.

Proof: Let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Let $V(K_m \square P_n) = \{(u_i, v_j)/1 \le i \le m, 1 \le j \le n\}$. deg (u_i, v_1) = deg $(u_i, v_n) = m$, $1 \le i \le m$. deg $(u_i, v_j) = m + 1$, $1 \le i \le m$, $2 \le j \le n - 1$. Suppose S is a strong efficient dominating set of $K_m \square P_n$. Suppose (u_i, v_2), $1 \le i \le m$, belongs to S. Then (u_i, v_2) strongly dominates (u_1, v_2), (u_2, v_2), …, (u_m, v_2), (u_i, v_1) and (u_i, v_3). To strongly dominate (u_k, v_1), $1 \le k \le m$, $k \ne i$, S must contain one of (u_k, v_1), $1 \le k \le m$, $k \ne i$. Hence (u_i, v_1) is strongly dominated by two elements (u_k, v_1) and (u_i, v_2) of S, a contradiction. Similarly we get a contradiction if any (u_i, v_j) belongs to S. Therefore $K_m \square P_n$ is not strong efficient. ■

Theorem 3.11: $K_m \square C_n$, $m \ge 1$, $n \ge 3$.

Proof: Let $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Let $V(K_m \square C_n) = \{(u_i, v_j)/1 \le i \le m, 1 \le j \le n\}$. deg$(u, v) = m + 1$ for all $(u, v) \in V(K_m \square C_n)$. Suppose S is a strong efficient dominating set of $K_m \square C_n$. Suppose (u_i, v_1), $1 \le i \le m$, belongs to S. Then (u_i, v_1) strongly dominates (u_1, v_1), (u_2, v_1), … , (u_m, v_1), (u_i, v_2) and (u_i, v_n). To strongly dominate (u_k, v_2), $1 \le k \le m$, $k \ne i$, S must contain one of (u_k, v_2), $1 \le k \le m$, $k \ne i$. Hence (u_i, v_2) is strongly dominated by two elements (u_k, v_2) and (u_i, v_1) of S, a contradiction. Similarly we get a contradiction if any (u_i, v_j) belongs to S. Therefore $K_m \square C_n$ is not strong efficient. ■

Theorem 3.12: $K_m \square K_n$, $m, n \ge 2$, is not strong efficient.

Proof: $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Let $V(K_m \square K_n) = \{(u_i, v_j)/1 \le i \le m, 1 \le j \le n\}$. deg$(u, v) = m + n - 2$ for all $(u, v) \in V(K_m \square K_n)$. Suppose S is a strong efficient dominating set of $K_m \square K_n$. Suppose (u_i, v_1), $1 \le i \le m$, belongs to S. Then (u_i, v_1) strongly dominates (u_1, v_1), (u_2, v_1), … , (u_m, v_1), (u_i, v_2), (u_i, v_3) … (u_i, v_n). Suppose any (u_j, v_k), $1 \le j \le m$, $j \ne i$, $1 \le k \le n$, $k \ne 1$ belongs to S then (u_j, v_1), $j \ne 1$ is adjacent with two elements (u_i, v_1) and (u_j, v_k) of S, a contradiction. Similarly we get a contradiction if any (u_i, v_j) belongs to S. Therefore $K_m \square K_n$ is not strong efficient. ■

Theorem 3.13: $K_m \square K_{1,n}$, $m \ge 1$, $n \ge 3$, is not strong efficient.

Proof: Let $V(K_m) = \{u_1, u_2 \ldots u_m\}$ and $V(K_{1,n}) = \{v_1, v_2 \ldots v_{n+1}\}$. Let $V(K_m \square K_{1,n}) = \{(u_i, v_j)/1 \le i \le m, 1 \le j \le n\}$. For $1 \le i \le m$, deg $(u_i, v_1) = n + m - 1 = \Delta(K_m \square K_{1,n})$ and deg$(u_i, v_j) = m$, $2 \le j \le n$. Let S be a strong efficient dominating set of $K_m \square K_{1,n}$. If any one of $\{(u_i, v_1), 1 \le i \le m\}$ belongs to S, then it strongly uniquely dominates (u_i, v_1) $1 \le i \le m$, and (u_i, v_k), $2 \le k \le n + 1$. If any one of $\{(u_j, v_k) / 1 \le j \le m\}$ belongs to S, then (u_1, v_k) is strongly dominated by two vertices (u_1, v_1) and (u_j, v_k), a contradiction. The argument is similar if any one of $\{(u_j, v_k) / 1 \le j \le m, 1 \le k \le n+1\}$. Therefore $K_m \square K_{1,n}$ is not strong efficient. ■

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